Computers and Mathematics with Applications 64 (2012) 2160–2165

Contents lists available at SciVerse ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Period two implies chaos for a class of multivalued maps: A naive approach

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ARTICLE INFO

Article history:
Received 9 June 2011
Received in revised form 22 November 2011
Accepted 22 November 2011

Keywords:
Chaos
Periodic orbits
Multivalued maps
Poincaré operators

ABSTRACT

On the background of our earlier results concerning the coexistence of infinitely many periodic orbits, we present a new theorem dealing with a large class of one-dimensional multivalued maps with monotone margins and connected values. If a nontrivial ($n > 1$) n-orbit occurs then, according to our theorem, these maps possess a single-valued chaotic selection, on a compact subinterval.

1. Introduction

The main aim of this paper is to demonstrate that there exists a large class of one-dimensional multivalued maps exhibiting chaos. These maps have a very strong forcing property. More concretely, if there exists a positive integer $n > 1$ such that they admit an n-orbit, then the coexistence of k-orbits occurs, for each $k \in \mathbb{N}$. At the same time, there also exists a single-valued continuous selection on a compact interval which is chaotic, practically in an arbitrary way. Roughly speaking, "*period two implies chaos*" here (whence the title). In fact, we will show that "*any nontrivial period is equivalent with many sorts of chaos*", but this requires to be explained in more detail.

In order to understand these phenomena in a deeper way, let us start with a survey of similar results for single-valued continuous functions. There are two classical results of this type, namely the Sharkovsky cycle coexistence theorem in [1] and the one due to Li and Yorke saying that "*period three implies chaos*" in [2]. The Sharkovsky theorem is based on a new (Sharkovsky) ordering of positive integers:
$3 \rhd 5 \rhd 7 \rhd 9 \rhd \cdots \rhd 2\cdot 3 \rhd 2\cdot 5 \rhd 2\cdot 7 \rhd 2\cdot 9 \rhd \cdots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5 \rhd 2^2\cdot 7 \rhd 2^2\cdot 9 \rhd \cdots 2^n\cdot 3 \rhd 2^n\cdot 5 \rhd 2^n\cdot 7 \rhd 2^n\cdot 9 \rhd \cdots \rhd 2^n \rhd \cdots \rhd 2^2 \rhd 2 \rhd 1.$

Theorem 1 (*Sharkovsky' 64*)**.** *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. If there is some $a \in \mathbb{R}$ which forms a primary n-orbit $\mathcal{O}$ of f, i.e.*

$$\mathcal{O} = \underbrace{\left\{a, f(a), \ldots, f^{n-1}(a)\right\}}_{\text{mutually different elements}}, \quad \text{where } f^n(a) = a,$$

then for each $k \lhd n$ (in the Sharkovsky ordering) there is, for some $b \in \mathbb{R}$, also a primary k-orbit of f.

Remark 1. The special case of Theorem 1, for $n = 3$, was obtained much later, but independently, in [2]. This forcing property is in principle one-dimensional, because the analogous criteria in $\mathbb{R}^n$ are very drastic, for $n > 1$ (cf. [3]).

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doi:10.1016/j.camwa.2011.11.043

Theorem 2 (*Li and Yorke, 1975*)**.** *Let a continuous function $f: \mathbb{R} \to \mathbb{R}$ admit a primary 3-orbit. Then there exists an uncountable subset $S \subset \mathbb{R}$ (called a scrambled set) with the following properties:*
(i) $\limsup_{m\to\infty} |f^m(x) - f^m(y)| > 0$, *for all* $x, y \in S$, $x \neq y$,
(ii) $\liminf_{m\to\infty} |f^m(x) - f^m(y)| = 0$, *for all* $x, y \in S$, $x \neq y$.

Remark 2. The properties (i), (ii) holding on a scrambled set $S \subset \mathbb{R}$ are nowadays called *chaos in the sense of Li–Yorke*.

Theorem 2 was extended in [4,5] to "*period* $\neq 2^n$ *implies chaos*". In [6], this implication was replaced by the equivalence, but the chaos had this time a different meaning, namely that f has a *positive topological entropy*, i.e. $h(f) > 0$. For its definition, see e.g. [7]. This is still equivalent with the existence of a *horseshoe* of f^k, for some $k \geq 1$, i.e. with the existence of an interval $I \subset \mathbb{R}$ and disjoint open subintervals K_1 and K_2 of I such that $f^k(K_1) = f^k(K_2) = I$, for some $k \geq 1$, (cf. [7]) as well as with the *topological transitivity* of f, i.e. with the existence of nonempty open subsets U and V of $\mathbb{R}$ such that $f^k(U) \cap V \neq 0$, for some $k \geq 1$. In fact, all the above equivalent properties coincide with the *chaos in the sense of Devaney* (see e.g. [8]), because the sole transitivity implies, on intervals in $\mathbb{R}$, that the set of periodic points of f is dense and that f has a sensitive dependence on initial conditions. Moreover, "*transitivity implies period 6*" (cf. [9,10]). For the related definitions and more details, see e.g. [11].

On the other hand, neither Theorem 1 nor Theorem 2 can be applied, via Poincaré's translation operators, to scalar ordinary differential equations. To be more precise, let us consider the scalar equation

$$x' = f(t, x), \tag{1}$$

where (for the sake of simplicity) $f \in C(\mathbb{R}^2, \mathbb{R})$ satisfies the linear growth restrictions

$$|f(t, x)| \leq \alpha + \beta|x|, \quad \text{for all } (t, x) \in \mathbb{R}^2,$$

and assume that

$$f(t, x) \equiv f(t + 1, x).$$

In view of Theorems 1 and 2, assume for a moment the unique solvability of (1). It is well-known (see e.g. [12, Theorem 9.1] that every bounded solution of (1) on the half-line is either 1-periodic or asymptotically 1-periodic which excludes the existence of subharmonic k-periodic solutions, for any $k > 1$. Another, even more transparent argument, concerns just the associated *Poincaré translation operators* $T_n: \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, along the trajectories of (1), i.e.

$$T_n(x_0) := \{x(n, x_0): x(\cdot, x_0) \text{ is a solution of (1), where } x(0, x_0) = x_0\}. \tag{2}$$

Because of a unique solvability of (1), solutions of (1) depend continuously on initial conditions by which T_n becomes completely continuous. Moreover, T_n is still strictly increasing (otherwise, a contradiction with uniqueness), and subsequently also homeomorphic. The monotonicity of T_n again excludes the existence of k-periodic points of T_n which determine in a one-to-one way k-periodic solutions of (1), for any $k > 1$.

Since, in the lack of uniqueness, the Poincaré operators are obviously multivalued, this was for us a stimulation to consider versions of Theorems 1 and 2 for multivalued maps, possibly applicable to differential equations and inclusions. Let us note that, according to the result of Orlicz, ordinary differential equations are generically uniquely solvable.

2. Sharkovsky-type theorems for multivalued maps

Hence, consider now multivalued maps $\varphi: \mathbb{R} \multimap \mathbb{R}$ (i.e. $\varphi: \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$).

Definition 1. By an *orbit of kth-order* (*k-orbit*) to a multivalued map $\varphi: \mathbb{R} \multimap \mathbb{R}$, we mean a sequence $\{x_i\}_{i=0}^{k-1}$ such that
(i) $x_{i+1} \in \varphi(x_i)$, $i = 0, 1, \ldots, k-2$,
(ii) $x_0 \in \varphi(x_{k-1})$,
(iii) this orbit is not a product orbit formed by going p-times around a shorter orbit of mth-order, where $mp = k$.

If still

(iv) $x_i \neq x_j$, for $i \neq j$; $i, j = 0, 1, \ldots, k-1$, then we speak about a *primary orbit of kth-order* (*primary k-orbit*).

Definition 2. A multivalued map $\varphi: \mathbb{R} \multimap \mathbb{R}$ is *upper semicontinuous* (u.s.c.) if, for any open $U \subset \mathbb{R}$, the small preimage $\{x \in \mathbb{R}: \varphi(x) \subset U\}$ of φ is also open.

For the Poincaré operators T_n in (2), their upper semicontinuity (which is always the case) is equivalent to the closedness of their graph Γ_{T_n}, where

$$\Gamma_{T_n} := \{(x, y) \in \mathbb{R}^2: x \in \mathbb{R},\ y \in T_n(x)\}.$$

Moreover, for every $x \in \mathbb{R}$, the set of values $\{T_n(x)\}$ of T_n consists either of a singleton or of a compact interval and $T_n^m = T_{nm}$, $m \in \mathbb{N}$. For more details, (see e.g. [13, Chapter III.4]).

For these maps, we were also able to prove the following theorem.

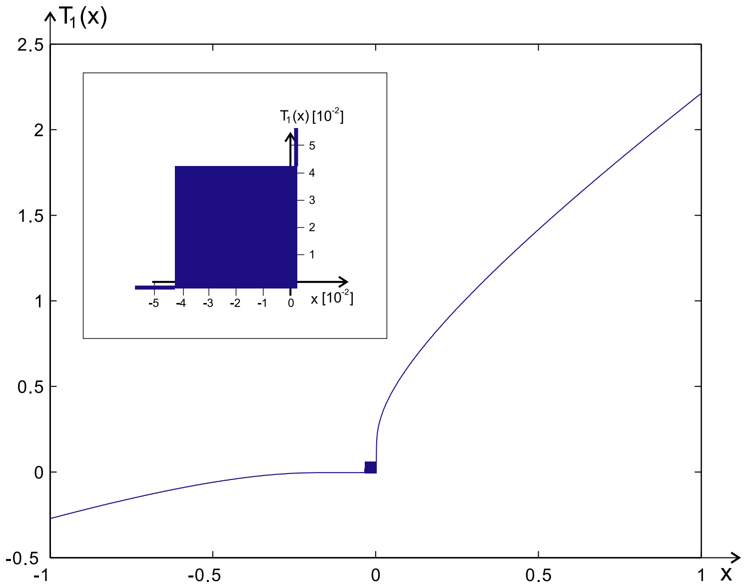

Fig. 1. Poincaré's operator T_1 to (3).

Theorem 3 (*cf. [14]*)**.** *Let $\varphi: \mathbb{R} \multimap \mathbb{R}$ be an upper semicontinuous map with convex compact values. Assume that φ has an n-orbit. Then φ admits, for each $k \lhd n$, also a k-orbit, with an eventual exception of at most two orbits.*

Remark 3. Theorem 3 cannot be improved because of counter-examples documented (see [13, Chapter III.4]). Since k-orbits of the Poincaré operators T_1 ($n = 1$) determine, in a not necessarily unique way, k-periodic solutions of (1), and vice versa, Theorem 3 can be immediately applied to differential equations of the form (1). Nevertheless, the exceptional cases remain also here.

In view of Remark 3, we were therefore rather surprised, when the paper [15] appeared.

Theorem 4 (*cf. [15]*)**.** *If Eq. (1) has, for some $n > 1$, an n-periodic solution then, for each $k \in \mathbb{N}$, it admits a k-periodic solution.*

Remark 4. Unlike in Theorem 3, Theorem 4 holds with no exceptions and, especially, there is no relationship to the Sharkovsky ordering.

The only explanation of two facts in Remark 4 is that the class of multivalued maps treated in Theorem 3 is still too wide for applications to Eq. (1) (see Fig. 4 below). In order to provide a better insight, let us present the following illustrative example.

Example 1. Equation

$$x' = \sqrt{|x|} - \frac{1}{8\pi}|\arcsin(\sin \pi t)| \tag{3}$$

admits, for each $k \in \mathbb{N}$, a k-periodic solution (cf. [16]).

The associated Poincaré translation operator T_1 to (3) is plotted in Fig. 1 (cf. [17]).

Observe that both the margins of T_1 are nondecreasing. It is clear that all important processes occur in the square of the zoom (see also Fig. 2). In fact, a smaller subsquare in a right corner below centered at the origin plays the dominant role here.

Definition 3. By the *margins* of a multivalued map $\varphi: \mathbb{R} \multimap \mathbb{R}$, we understand the single-valued functions $\varphi^*: \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ and $\varphi_*: \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\varphi^*(x) := \sup\{y : y \in \varphi(x)\}, \qquad \varphi_*(x) := \inf\{y : y \in \varphi(x)\}.$$

Having this in mind, we were able to prove the following theorem.

Theorem 5 (*cf. [17,18]*)**.** *Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a multivalued mapping with nonempty connected values whose margins are either both nondecreasing or both nonincreasing. If φ has an n-orbit with $n > 1$, then it also admits a primary k-orbit, for each $k \in \mathbb{N}$.*

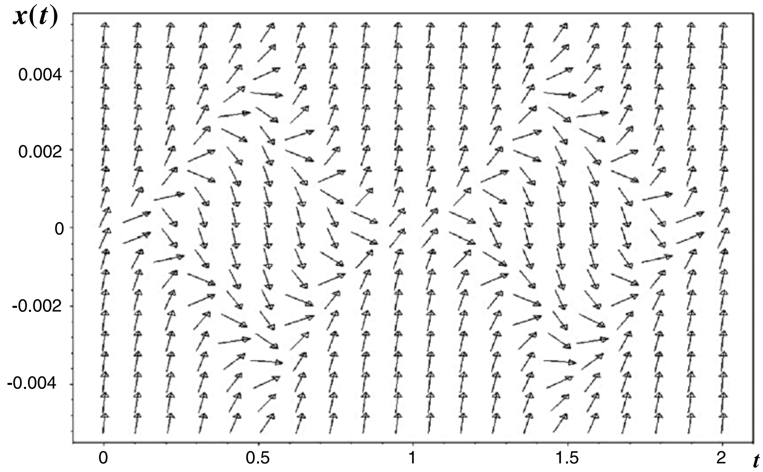

Fig. 2. Generating vector field for (3).

Corollary 1. *Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a multivalued mapping with nonempty connected values whose margins are either both nondecreasing or both nonincreasing. If φ possesses, on a compact interval $I \subset \mathbb{R}$, a single-valued continuous selection $s \subset \varphi|_I$, $s: I \to I$, which is topologically transitive, then the coexistence of primary k-orbits of φ occurs, for each $k \in \mathbb{N}$.*

Proof. Since one-dimensional transitive functions admit 6-orbits (cf. [9,10]), so does φ, and Theorem 5 applies. □

3. Li–Yorke type theorem for multivalued maps

At this moment, there are to our disposal four alternative proofs of Theorem 4:

- in [15], by means of an upper and lower solutions technique,
- in [17,18], via multivalued Poincaré's translation operators (i.e. by means of Theorem 5),
- in [19], via direct geometric considerations,
- in [20], in the frame of dynamical systems.

The imitation in [15] of the celebrated title of [2] was motivated by the observation made in [15, Remark 2] that, in a suitable sense, every possible discrete dynamics can be realized by a solution of (1). Later, using more sophisticated techniques, Pireddu [20] formalized this fact, proving for (1), under the assumptions of Theorem 4, the existence of chaos in the sense of Devaney, Li–Yorke, coin-tossing and a positive topological entropy.

In contrast to [20], our Theorem 6 dealing with a significantly larger class of related multivalued maps (see Fig. 4 below) will be proved in an extremely simple way.

Theorem 6. *Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a multivalued mapping with nonempty connected values whose margins are either both nondecreasing or both nonincreasing. If φ has an n-orbit with $n > 1$, then φ is chaotic in the sense that there exists a compact interval $I \subset \mathbb{R}$ on which $\varphi|_I$ possesses a chaotic single-valued continuous selection $s \subset \varphi$, $s: I \to I$. The chaos can be understood here in an arbitrary sense (Devaney, Li–York, a positive topological entropy, a horseshoe, etc.).*

Proof. Assume the margins φ^*, φ_* are nondecreasing, the other case being treated similarly. Let $\{x_0, \ldots, x_{n-1}\}$ be the considered n-orbit. Redenote the points as $\{y_0, \ldots, y_{n-1}\}$ so that y_0 is the minimal element. Observe that there exists $y_i \in \{y_0, \ldots, y_{n-1}\}$ such that $y_i > y_0$ and $y_0 \in \varphi(y_i)$ (otherwise, the n-orbit would be reduced to the single point y_0). The monotonicity of φ_* now implies that $y_0 \in \varphi(y_0)$.

Observe further that due to the connectedness of the values of φ, the set $\varphi(y_0)$ contains a nondegenerate interval, say $[y_0, a]$ (otherwise, the n-orbit would be again reduced to the single point y_0). We may take $a < y_i$. Summing up the above arguments, the whole square $[y_0, a]^2$ is contained in the graph Γ_φ of φ because of the monotonicity of the margins (see Fig. 3).

One can easily check that the graph of, for instance, the tent map $T: [y_0, a] \to [y_0, a]$, where

$$T(x) := \begin{cases} 2x - y_0, & \text{for } y_0 \le x < \frac{1}{2}(y_0 + a), \\ 2(a - x) + y_0, & \text{for } \frac{1}{2}(y_0 + a) \le x < a, \end{cases}$$

is contained in the given square, i.e. $\Gamma_T \subset [y_0, a]^2 \subset \Gamma_\varphi$ (see again Fig. 3).

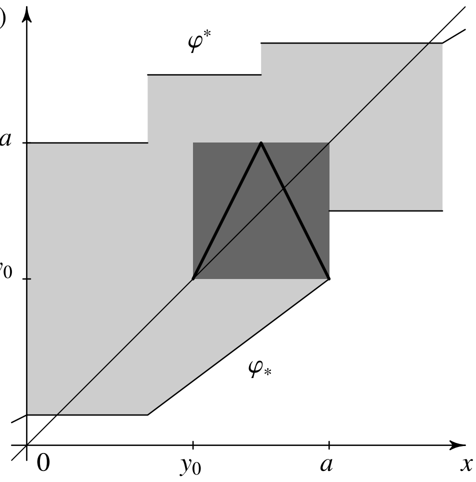

Fig. 3. Square $[y_0, a]^2 \subset \Gamma_\varphi$ containing the graph Γ_T of the tent map T.

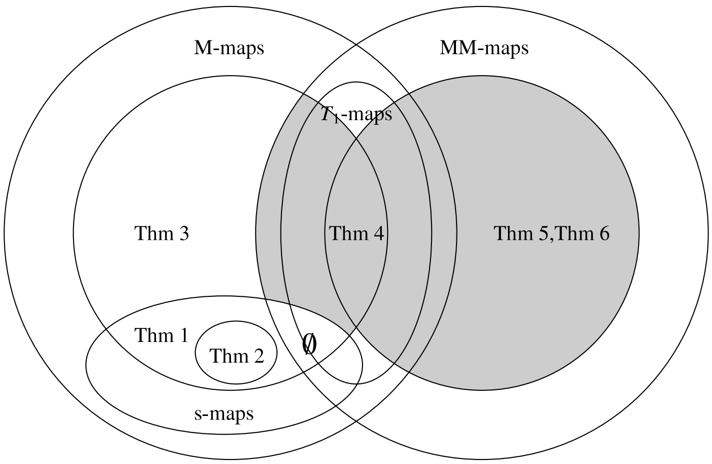

Fig. 4. Venn's diagram of map classes, where M-maps . . . u.s.c. maps with convex compact values, MM-maps . . . convex-valued maps with monotone margins, s-maps . . . single-valued continuous functions, T_1 . . . Poincaré's operators, Thm i . . . maps concerned in Theorem i, $i = 1, \ldots, 6$.

Since the tent map T is well-known to be chaotic in the sense that T^2 has a *horseshoe* (see e.g. [7, Lemma 11.20]), its *topological entropy* $h(T)$ must be positive (see e.g. [7, Lemma 11.7]), i.e. $h(T) > 0$. This is, according to [6], still equivalent to the existence of an orbit with period different to 2^n, and subsequently to the chaos in the sense of Devaney (see e.g. [8]). More direct arguments for the Devaney chaos can be found e.g. in [11]. In particular, it must be also chaotic in the sense of Li–York (see e.g. [8]), etc. □

Since the margins of the Poincaré operators are nondecreasing (see [17,18]), we can immediately give the following corollary which is very similar to the result of Pireddu in [20].

Corollary 2. *If Eq. (1) has, for some $n > 1$, an n-periodic solution, then the associated Poincaré operator T_1 along the trajectories of (1), defined in (2), possesses, on a compact interval, a single-valued continuous chaotic selection.*

4. Concluding remarks

In fact, there exist continua of chaotic single-valued selections of multivalued maps under consideration in Theorem 6. We made a choice of the most standard one because of its simplicity.

By Definition 3, the margins or their parts need not belong to the graphs of multivalued maps. Moreover, the maps in Theorem 6 need neither be upper semicontinuous nor lower semicontinuous.

Observe that the map in Fig. 3 is not lower semicontinuous, but the closure (if necessary) of its graph belongs to an upper semicontinuous map.

A similar situation can be detected for an upper semicontinuous mapping in Fig. 2, but for the optical evidence one would have to make a further zoom of the given one (i.e. a zoom of the zoom). The square on which a chaotic selection (e.g. again a tent map) exists would be approximately $[-2 \cdot 10^{-3}, 2 \cdot 10^{-3}]^2$.

Theorems 5 and 6 can be still slightly improved in the sense that the margins of multivalued maps can be only monotone on a dense subset of $\mathbb{R}$ (cf. [18]). Corollary 1 represents, in view of Theorem 5, a particular reverse implication to Theorem 6. Thus, for one-dimensional convex-valued maps with monotone margins, the existence of a transitive selection $s \subset \varphi|_I$ of φ is equivalent with the coexistence of primary k-orbits of φ, for each $k \in \mathbb{N}$. Nevertheless, to say simply that "*any period* >1 *is equivalent with chaos*" would be only correct, when speaking here exclusively in terms of periodic orbits, but not of periodic points (cf. [13]).

The class of ordinary differential equation (1) is, in view of the mentioned Orlicz generic theorem, rather narrow w.r.t. the applications of Theorem 6 (i.e. in Corollary 2). On the other hand, since the Poincaré translation operators along the trajectories of upper-Carathéodory differential inclusions have the same regularity properties, and thus belong to the same class, the application of Theorem 6 to multivalued differential equations (inclusions) would not be exceptional. The same is true for Theorem 5 and Corollary 1.

In order to have a more concrete idea about the applicability of given theorems, let us finally see Fig. 4, where the mutual relationship of map classes under consideration is presented.

Acknowledgment

The work was supported by the Council of Czech Government (MSM 6198959214).

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